Contribution to the theory of cellular thermal convection

By ENOK PALM† AND HENRY ØIANN
Institute of Mechanics, The Technical University of Norway, Trondheim

(Received 11 February 1963 and in revised form 3 February 1964)

In a previous paper on cellular thermal convection (Palm 1960) the importance of the effect caused by temperature variation of kinematic viscosity was pointed out. It was demonstrated that this effect would, owing to non-linear interactions, lead to a tendency towards hexagonal cells. For mathematical simplicity, only the interaction of two wave-components was taken into account.

Segel & Stuart (1962), working with the same equations, have examined the stability of the various equilibrium solutions. They arrive at the important conclusion that a necessary condition for the solution corresponding to hexagons to be stable is that the variation of viscosity with temperature be sufficiently great.

In the present paper the problem is discussed from a somewhat more general point of view. First it is shown that, when the variation of viscosity with temperature is sufficiently great, the solution corresponding to hexagons is the *only* stable one if only two wave-components are taken into account. To examine if this result is also true when the motion consists of an arbitrary number of wave-components, the case of three wave-components is studied. It turns out that in this case also the only possible mode is the pattern consisting of hexagons. The validity of this result is easily extended to a more general class of wave-components. It is shown that the solution corresponding to hexagons is stable for all small disturbances which can possibly occur. To prove this it is necessary to take into account non-linear disturbance theory.

A reasonable conclusion from the paper by Segel & Stuart and the present paper is that a hexagonal pattern is observed only when a condition of the form (6.9) is fulfilled. Experiments concerning this problem are, however, lacking.

1. Introduction

In a recent paper on cellular thermal convection (Palm 1960), the problem of why hexagons are preferred cells was discussed. As is well known, the linearized equations do not give any answer to this problem: when the critical Rayleigh number is attained, all waves satisfying

$$k^2+l^2 = r^2 \tag{1.1}$$

start to grow according to linear theory. Here k and l are the wave-numbers in the horizontal x- and y-directions, respectively, and r^2 is a quantity depending

† Now at Institute of Mechanics, University of Oslo.

on the nature of the boundary conditions. For 'free' boundaries (characterized by zero vertical velocity, zero shear stress and specified temperature), as was assumed in the paper referred to,

$$r^2 = \tfrac{1}{2}(\pi/h)^2 = \tfrac{1}{2}\lambda^2, \tag{1.2}$$

with h denoting the depth of the fluid layer. Hence, as long as the linearized equations govern the motion, this will be very complex and display no sign of regularity. It is therefore obvious that for the problem in question non-linear equations must be taken into account.

In order to give a complete solution of the problem, the initial motion should have been composed of an arbitrary distribution of waves satisfying (1.1). This leads to a very complicated mathematical problem, and, for mathematical simplicity, it was assumed in the paper mentioned above that the initial motion (i.e. the initial value of the vertical velocity) consisted *mainly* of one two-dimensional Fourier-component

$$A_{021}\cos 2ly \sin\lambda z \quad (4l^2 = r^2). \tag{1.3}$$

In addition, the initial motion also consisted of the other waves satisfying (1.1), this part of the wave-spectrum, however, being of much smaller amplitude. The problem then arose as to whether it was possible to find any non-linear mechanism which would change the initial (approximate two-dimensional) motion into a hexagonal cellular pattern. Hexagons are composed of the two Fourier-components

$$A_{11}\cos kx \cos ly + A_{02}\cos 2ly \quad (k^2 = 3l^2), \tag{1.4}$$

where

$$A_{11} = 2A_{02}. \tag{1.5}$$

Thus the problem was reduced to investigating whether a (non-linear) coupling term existed which would lead to the wave-component

$$A_{111}(t)\cos kx \cos ly \sin\lambda z \quad (k^2 = 3l^2), \tag{1.6}$$

being the only preferred component of the group (1.1), such that $A_{111} \to 2A_{021}$ as $t \to \infty$.

It turned out that such a coupling term existed if the effect of the variation of viscosity with temperature was taken into account. The kinematic viscosity ν was written in the form

$$\nu = \nu_0 + \gamma\cos[(\lambda/\beta)(\Theta+\theta)] = \nu_0 + \gamma\cos\lambda z + (\gamma\lambda/\beta)\,\theta\sin\lambda z, \tag{1.7}$$

where $\gamma/\nu_0 \ll 1$. β denotes the initial uniform temperature gradient, $\Theta(z,t)$ the mean temperature, and $\theta(x,y,z,t)$ the departure of the temperature from this mean. The differential equations for the amplitudes $A_{111}(t)$ and $A_{021}(t)$ were then found to be, correct to the third-order terms,†

$$K\dot{A}_{111} = \epsilon A_{111} - aA_{111}A_{021} - RA_{111}^3 - PA_{111}A_{021}^2, \tag{1.8}$$

$$K\dot{A}_{021} = \epsilon A_{021} - \tfrac{1}{4}aA_{111}^2 - R_1A_{021}^3 - \tfrac{1}{2}PA_{111}^2A_{021}. \tag{1.9}$$

† The coefficients of the third-order terms in (1.9) were incorrectly given in Palm (1960). They are given in a corrected form in the paper by Segel & Stuart (1962). The notation applied here diverges slightly from that applied in their paper.

Here overdots denote differentiation with respect to time, κ is the thermal diffusivity,

$$\epsilon = g\alpha\Delta\beta r^2, \tag{1.10}$$

$$\alpha = (\lambda\gamma/8)(3r^4 - r^2\lambda^2 + 3\lambda^4) \tag{1.11}$$

is the coefficient of expansion, $\Delta\beta$ is the difference between the actual initial temperature gradient β and the critical initial temperature gradient for the onset of convection β_0. $\Delta\beta/\beta_0$ is assumed small compared to unity,

$$R = R(k,l) = \frac{\lambda\kappa\sigma^2}{2r^2}[l^2P_1 + \tfrac{1}{4}k^2G/M] + \frac{g\alpha\beta_0\lambda}{32\kappa}\left[\frac{l^2(4P_1 + l^2\lambda/\kappa\sigma r^2)}{k^2+\lambda^2} + \frac{k^2(G/M + k^2\lambda/\kappa\sigma r^2)}{l^2+\lambda^2} + \frac{2r^2}{\lambda\kappa\sigma}\right], \tag{1.12}$$

$$R_1 = \tfrac{1}{8}g\alpha\beta_0 r^2/\kappa^2\sigma, \tag{1.13}$$

$$P = 4R - R_1, \tag{1.14}$$

$$P_1 = \frac{\lambda k^2 l^2}{r^4}\frac{\kappa\sigma(k^2+\lambda^2) + \beta g\alpha r^2/4\kappa\sigma}{16\kappa\nu_0(k^2+\lambda^2)^3 - g\alpha\beta_0 k^2}, \tag{1.15}$$

$$G = (16\lambda k^2 l^2/r^4)[\kappa\sigma(l^2+\lambda^2) + \beta g\alpha r^2/4\kappa\sigma], \tag{1.16}$$

$$M = 64\kappa\nu_0(l^2+\lambda^2)^3 - 4g\alpha\beta_0 l^2, \tag{1.17}$$

$$\sigma^2 = k^2 + l^2 + \lambda^2, \tag{1.18}$$

$$K = (\kappa + \nu_0)\sigma^2. \tag{1.19}$$

We notice that the two terms in equations (1.8), (1.9),

$$aA_{111}A_{021} \quad \text{and} \quad \tfrac{1}{4}aA_{111}^2 \tag{1.20}$$

are coupling terms which are destabilizing when

$$aA_{021} < 0. \tag{1.21}$$

It is seen that only wave-components for which $k^2 = 3l^2$ give rise to such terms. The terms (1.20) are therefore of the required type. Since both of the terms (1.20) are destabilizing when (1.21) is fulfilled, the destabilizing effect is mutual. In order that the final motion shall have a hexagonal pattern, (1.5) must be fulfilled when $t \to \infty$. It is readily seen that (1.8), (1.9) really contain such a stationary solution. ((1.8), (1.9) also have other stationary solutions, as pointed out by Segel & Stuart 1962. This will be discussed later in the paper.)

To investigate the meaning of the relation (1.21), we first mention that if (1.21) is fulfilled initially, it is also fulfilled for subsequent times, as seen from (1.9). Therefore, by choosing the frame of reference properly, we are always able to ensure that (1.21) is satisfied for all times. This relation determines the sign of A_{021} and therefore the direction of circulation in the hexagons. Hence, for fluids in which ν increases with temperature (a positive) the flow descends in the middle of the cell and for fluids in which ν decreases with temperature (a negative) the flow ascends in the middle of the cell, in accordance with experiments.

It is of interest to note that if the frame of reference is displaced a distance $\frac{1}{2}\pi l^{-1}$ along the y-axis, the results given above imply that the wave-component (1.3) and the wave-component

$$A_{111}(t)\cos kx \sin ly \sin \lambda z \tag{1.22}$$

are mutually destablizing for all times when

$$aA_{021} > 0. \tag{1.23}$$

Hence, if the frame of reference is chosen *beforehand*, the wave-components (1.3) and (1.6) reinforce each other for all times when (1.21) is fulfilled, whereas the wave-components (1.3) and (1.22) reinforce each other for all times when (1.23) is fulfilled. This matter will be discussed somewhat more in the next section.

Before proceeding to the next section, it should be pointed out that only the variation of ν due to the motion (i.e. the term $(\gamma\lambda/\beta)\theta\sin\lambda z$ in (1.7)) gives rise to the terms proportional to a in (1.8) and (1.9).

2. Comments on the initial motion

In Segel & Stuart (1962), the various equilibrium solutions of (1.8), (1.9) are discussed, and the stability of these solutions are examined. We now summarize their principal results concerning the stability of the equilibrium solutions:

(i) The equilibrium solution corresponding to hexagons is stable for infinitesimal disturbances when (1.21) is fulfilled and a as sufficiently large. More precisely, the condition that this solution be stable is that

$$a^2\epsilon^{-1} > 4(2R-R_1)^2/(4R+R_1). \tag{2.1}$$

(ii) Even though this condition is satisfied, the solution corresponding to hexagons is not the only one. The equations also possess a stable equilibrium solution corresponding to two-dimensional rolls in this domain.

We shall, however, show in the next section that this last result is incorrect. It may therefore be concluded that the solution corresponding to hexagons is the *only* stable solution when $a^2\epsilon^{-1}$ is sufficiently large.

In order to get some information about what determines the location of the hexagons, we return to a discussion of the initial motion chosen above. It is evident that if, instead of the wave-component (1.6), we consider the wave-component

$$B_{111}\sin kx\cos ly\sin\lambda z \quad (k^2 = 3l^2), \tag{2.2}$$

the final motion will consist of hexagons displaced a distance $\frac{1}{2}\pi k^{-1}$ along the x-axis compared to the first system of hexagons. In a real fluid both the wave-components (1.6) and (2.2) are present, and it is readily shown that the final motion will consist of hexagons, the centres of which are determined by the relative magnitudes of the initial values of the amplitudes A_{111} and B_{111}. If, instead of (1.6), we consider the wave-component

$$C_{111}\cos kx\sin ly\sin\lambda z \quad (k^2 = 3l^2), \tag{2.3}$$

we obtain, as is easily seen by displacing the frame of reference a distance $\frac{1}{2}\pi l^{-1}$ in the y-direction, the same equations as (1.8) and (1.9) except that the terms proportional to a change sign. We may without any loss of generality assume that aA_{021} is initially negative. The two terms (1.20) are then initially stabilizing. However, if $a^2\epsilon^{-1}$ is sufficiently large, the second terms on the right in (1.8), (1.9) determine the development of A_{111} and A_{021}, and it is readily seen that in this case A_{021} changes sign. We therefore end up with hexagons satisfying (1.23).

This may also be concluded from figure 2 in the paper by Segel & Stuart. The hexagons obtained by this wave-component and (1.3) are displaced a distance $\frac{1}{2}\pi l^{-1}$ along the y-axis compared to those defined by (1.6) and (1.3). If both the wave-components (1.6), (2.3) are present simultaneously, it may be shown that the final motion will be hexagons, located either as those determined by (1.6) and (1.3) or as those determined by (2.3) and (1.3), depending on the relative magnitude of the initial values of the amplitudes.

An arbitrary wave-component of the group (1.1),

$$A'_{111}(t)\cos k'x \cos l'y \sin \lambda z \quad (k'^2 \neq 3l'^2), \tag{2.4}$$

does not combine with (1.3) in such a way that the equations contain terms proportional to a. If all the three wave-components (1.3), (1.6), (2.4) are present simultaneously, it is found that the development of $A'_{111}(t)$ is governed by the equation

$$K\dot{A}'_{111} = \epsilon A'_{111} - R'A'^3_{111} - SA^2_{111}A'_{111} - TA^2_{021}A'_{111}. \tag{2.5}$$

Here

$$R' = R(k', l'), \tag{2.6}$$

$$S = S(k,l) + S(-k,l) + S(k,-l) + S(-k,-l) + R_1, \tag{2.7}$$

where
$$S(k,l) = \frac{(r^2 - kk' - ll')^2\lambda^2}{32r^4}\left[\frac{U}{V}\left(\frac{g\alpha\beta}{\kappa(\sigma+\lambda^2+kk'+ll')} + \frac{2\kappa\sigma^2}{r^2}\right) + \frac{g\alpha\beta r^2}{2\sigma\kappa^2(\sigma+\lambda^2+kk'+ll')}\right],$$

with

$$U = \kappa\sigma(\sigma+\lambda^2+kk'+ll') + \alpha\beta g r^2/2\kappa\sigma$$

and

$$\left.\begin{aligned} V &= 4\kappa\nu(\sigma+\lambda^2+kk'+ll')^3/(r^2+kk'+ll') - \alpha\beta g,\\ T &= 4S(0,2l) + 4S(0,-2l) + R_1. \end{aligned}\right\} \tag{2.8}$$

It is noted that all non-linear terms are stabilizing. The presence of the wave-component (2.4) will result in a term

$$-SA'^2_{111}A_{111} \tag{2.9}$$

being added on the right in (1.8), and in a term

$$-QA'^2_{111}A_{111} \tag{2.10}$$

being added on the right in (1.9). Here Q is given by

$$Q = \tfrac{1}{2}T_0. \tag{2.11}$$

3. Comments on the paper by Segel & Stuart

As mentioned above, Segel & Stuart show that a necessary condition for the solution corresponding to hexagons to be stable, is that the variation of kinematic viscosity with temperature be sufficiently large (relation (2.1)). They find, however, that equations (1.8), (1.9) also have other stable solutions in this region. It is easily seen that the two-dimensional motions

$$A_{111} = 0, \quad A_{021} = \pm(\epsilon/R_1)^{\frac{1}{2}} \tag{3.1 a, b}$$

are equilibrium solutions. According to Segel & Stuart, (3.1 a) is stable when

$$a\epsilon^{-\frac{1}{2}} > -2(2R - R_1)/R_1^{\frac{1}{2}} \tag{3.2 a}$$

and (3.1 *b*) is stable when

$$a\epsilon^{-\frac{1}{2}} < 2(2R-R_1)/R_1^{\frac{1}{2}}. \qquad (3.2b)$$

Hence, even for large values of the variation of viscosity with temperature, a two-dimensional stable solution also exists. As will now be demonstrated, this result is not correct.

Segel & Stuart examine the stability of the equilibrium solutions of (1.8), (1.9) by introducing an infinitesimal disturbance of the form

$$\delta A_{111} \cos kx \cos ly \sin \lambda z + \delta A_{021} \cos 2ly \sin \lambda z. \qquad (3.3)$$

They prove that the solution (3.1 *a*) is unstable for disturbances of the form (3.3) when

$$a\epsilon^{-\frac{1}{2}} < -2(2R-R_1)/R_1^{\frac{1}{2}}. \qquad (3.4)$$

However, if we displace the frame of reference a distance $\frac{1}{2}\pi l^{-1}$ along the y-axis, the solution (3.1 *a*) changes sign (3.1 *a* → 3.1 *b*), and the disturbance (3.3) takes the form

$$\delta A_{111} \cos kx \sin ly \sin \lambda z - \delta A_{021} \cos 2ly \sin \lambda z. \qquad (3.5)$$

We may therefore conclude that the solution (3.1 *b*) is *unstable* for perturbations of the form (3.5) when (3.4) is fulfilled. Correspondingly, we find that the solution (3.1 *a*) is *unstable* for perturbations of the form (3.5) when

$$a\epsilon^{-\frac{1}{2}} > 2(2R-R_1)/R_1^{\frac{1}{2}}. \qquad (3.6)$$

Hence, the two-dimensional equilibrium solutions (3.1 *a*, *b*) are *unstable* when

$$a^2\epsilon^{-1} > 4(2R-R_1)^2/R_1. \qquad (3.7)$$

Therefore, by taking into account a disturbance of form (3.5) in addition to (3.3), we obtain a modification of the result found by Segel & Stuart. We notice that in the case of sufficiently large variations of kinematic viscosity with temperature (relation (3.7)), hexagons are the *only* stable equilibrium solution.

The finding of Segel & Stuart that more than one stable equilibrium solution exist, seems, however, to be correct when

$$a^2\epsilon^{-1} < 4(2R-R_1)^2/R_1. \qquad (3.8)$$

We have so far only considered the case when the equilibrium solution consists of *two* wave-components. The important question is whether the result found above is also valid in the general case when the equilibrium solution consists of an infinite number of wave-components satisfying (1.1). It may first be noted that if $a = 0$ (kinematic viscosity independent of temperature), an infinite number of stable equilibrium solutions exist. It is therefore evident that if $a^2\epsilon^{-1}$ is sufficiently small, the final motion is composed of an infinite number of modes. To get some more information about the final motion when $a^2\epsilon^{-1}$ is large, we shall in the next section take into account also an arbitrary third wave-component, viz. (2.4). It will turn out that in this case also the equilibrium solution corresponding to hexagons is the *only* stable one. In § 5 we shall demonstrate that this equilibrium solution is also stable for arbitrary infinitesimal disturbances.

4. The equilibrium solution composed of three wave-components

The next step towards a more general investigation of the stability problem is to examine the stability of the equilibrium solutions when the three wave-components (1.3), (1.6) and (2.4) are present. The equations determining these wave-components are

$$K\dot{A}_{111} = \epsilon A_{111} - aA_{111}A_{021} - RA_{111}^3 - PA_{111}A_{021}^2 - SA_{111}'^2A_{111}, \tag{4.1}$$

$$K\dot{A}_{021} = \epsilon A_{021} - \tfrac{1}{4}aA_{111}^2 - R_1A_{021}^3 - \tfrac{1}{2}PA_{111}^2A_{021} - QA_{111}'^2A_{021}, \tag{4.2}$$

$$K\dot{A}_{111}' = \epsilon A_{111}' - R'A_{111}'^3 - SA_{111}^2A_{111}' - TA_{021}^2A_{111}'. \tag{4.3}$$

Let us temporarily disregard the solutions corresponding to

$$A_{111} = 0, \tag{4.4}$$

or

$$A_{111}' = 0. \tag{4.5}$$

The other equilibrium solutions are determined by

$$\epsilon - aA_{021} - RA_{111}^2 - PA_{021}^2 - SA_{111}'^2 = 0, \tag{4.6}$$

$$\epsilon A_{021} - \tfrac{1}{4}aA_{111}^2 - R_1A_{021}^3 - \tfrac{1}{2}PA_{111}^2A_{021} - QA_{111}'^2A_{021} = 0, \tag{4.7}$$

$$\epsilon - R'A_{111}'^2 - SA_{111}^2 - TA_{021}^2 = 0. \tag{4.8}$$

Eliminating A_{111} and A_{111}', we obtain

$$AA_{021}^3 + BaA_{021}^2 + Ca^2A_{021} + Da\epsilon = 0, \tag{4.9}$$

where

$$A = R_1(S^2 - RR') + \tfrac{1}{2}P(PR' - ST) - Q(PS - RT), \tag{4.10}$$

$$B = \tfrac{3}{4}PR' - \tfrac{1}{4}ST - QS, \tag{4.11}$$

$$C = \tfrac{1}{4}R' + \epsilon a^{-2}[RR' - S^2 + \tfrac{1}{2}P(S - R') + Q(S - R)], \tag{4.12}$$

$$D = \tfrac{1}{4}(S - R'). \tag{4.13}$$

For sufficiently large values of $a^2\epsilon^{-1}$, (4.9) has two solutions proportional to a. According to (4.8), the corresponding values of A_{111} and A_{111}' are imaginary, and these solutions must be rejected. The third solution is given by

$$\tfrac{1}{4}a^2A_{021} - \tfrac{1}{4}a(1 - S/R')\epsilon = 0. \tag{4.14}$$

Hence,

$$A_{021} = L\epsilon a^{-1}, \tag{4.15}$$

where

$$L = (1 - S/R'). \tag{4.16}$$

The corresponding values of A_{111} and A_{111}' are found to be

$$A_{111} = \pm 2(1 - Q/R')L^{\frac{1}{2}}\epsilon a^{-1}, \tag{4.17}$$

$$A_{111}' = \pm(\epsilon/R')^{\frac{1}{2}}. \tag{4.18}$$

To investigate the stability of this solution, we introduce the infinitesimal disturbances δA_{111}, δA_{021}, $\delta A_{111}'$. We obtain from (4.1), (4.2) and (4.3), applying the equilibrium equations,

$$K\delta\dot{A}_{111} = -2RA_{111}^2\delta A_{111} - (a + 2PA_{021})A_{111}\delta A_{021} - 2SA_{111}A_{111}'\delta A_{111}', \tag{4.19}$$

$$K\delta\dot{A}_{021} = -(\tfrac{1}{2}a + PA_{021})A_{111}\delta A_{111} + (\tfrac{1}{4}aA_{111}^2/A_{021} - 2R_1A_{021}^2)\delta A_{021} - 2QA_{021}A_{111}'\delta A_{111}', \tag{4.20}$$

$$K\delta\dot{A}_{111}' = -2SA_{111}A_{111}'\delta A_{111} - 2TA_{021}A_{111}'\delta A_{021} - 2R'A_{111}'\delta A_{111}'. \tag{4.21}$$

The disturbances have a time factor $e^{\sigma t}$, which introduced into these equations leads to the following third-order equation in σ:

$$\begin{vmatrix} -\sigma-2RA_{111}^2 & -(a+2PA_{021})A_{111} & -2SA'_{111}A_{111} \\ -(\tfrac{1}{2}a+PA_{021})A_{111} & -\sigma+\tfrac{1}{4}aA_{111}^2/A_{021}-2R_1A_{021}^2 & -2QA'_{111}A_{021} \\ -2SA_{111}A'_{111} & -2TA_{021}A'_{111} & -\sigma-2R'A'^2_{111} \end{vmatrix} = 0. \quad (4.22)$$

Algebraic equations of arbitrary order have been studied extensively in the literature in order to find the necessary and sufficient conditions for all roots to have negative real parts (stable solutions). In the case of a third-order equation

$$\sigma^3+a_1\sigma^2+a_2\sigma+a_3 = 0, \quad (4.23)$$

the Routh–Hurwitz condition (see Gantmacher 1959, pp. 226–33) gives that the necessary and sufficient conditions for only stable solutions are

$$a_1 > 0, \quad (a_1a_2-a_3) > 0, \quad a_3 > 0. \quad (4.24)$$

The last condition shows that the determinant, obtained from (4.22) by equating σ to zero, must be negative. For sufficiently large values of $a^2\epsilon^{-1}$ this determinant reduces to

$$R'A_{111}^2A'^2_{111}a^2, \quad (4.25)$$

which is a positive quantity. Hence, the solution (4.15) is unstable.

Correspondingly it may be shown that solutions satisfying (4.4) are unstable. In the special case when

$$A_{111} = A_{021} = 0, \quad A'_{111} = \pm(\epsilon/R')^{\frac{1}{2}}, \quad (4.26)$$

the motion is stable for disturbances of the type considered above. To prove the instability of the solution (4.26), we therefore have to introduce another set of disturbances. Let us write

$$k' = r\cos\phi, \quad l' = r\sin\phi. \quad (4.27)$$

We then consider a disturbance of the form

$$\delta A''_{111}\cos k''x\cos l''y\sin\lambda z+\delta A'''_{111}\cos k'''x\cos l'''y\sin\lambda z, \quad (4.28)$$

where $$k'' = r\cos(\phi-\tfrac{1}{3}\pi), \quad l'' = r\sin(\phi-\tfrac{1}{3}\pi), \quad (4.29)$$

and $$k''' = k'-k'', \quad l''' = l'-l''. \quad (4.30)$$

It is noted that $$(k'')^2+(l'')^2 = (k''')^2+(l''')^2 = r^2. \quad (4.31)$$

For large values of $a^2\epsilon^{-1}$ the equation determining A'''_{111} obviously must be of the form

$$K\delta\dot{A}'''_{111} = -LA'_{111}\delta A''_{111}. \quad (4.32)$$

It is found after some computation that

$$L = \tfrac{1}{2}a. \quad (4.33)$$

Since a is only a function of r^2 and λ^2 (and not explicitly of the wave-number), we correspondingly find that

$$K\delta\dot{A}''_{111} = -LA'_{111}\delta A'''_{111}. \quad (4.34)$$

Equations (4.32) and (4.34) reveal that the solution (4.26) is unstable. Hence, a pattern consisting of a regular system of rectangles is unstable.

Thus we arrive at the result that the only possible stable solutions are those satisfying (4.5). These solutions were discussed in the previous section. It will be shown in the next section that these solutions are also stable for perturbations of the type $\delta A'_{111}$.

It may be of interest to mention that Chandrasekhar (1961) points out that an expression of the form

$$w = \sin\lambda z\left\{A\cos\left[r\left(1-\frac{1}{m^2}\right)^{\frac{1}{2}}x\right]\cos\frac{r}{m}y + B\cos r(y-y_0)\right\}, \tag{4.35}$$

where m is an integer greater than one, appears to be the solution of the linearized problem giving the most general cell patterns which exhibit definite periodicities in the x- and y-directions. It is noted that $A = 2$, $m = 2$, $y_0 = 0$ correspond to hexagons.

It is seen from (4.2) and (4.3) (with $A = A'_{111}, B = A_{021}, A_{111} = \dot{A}_{021} = \dot{A}_{111} = 0$) that both A and B are of order ϵ. Introducing a disturbance of the form

$$\delta A_{111}\cos kx\cos l(y-y_0)\sin\lambda z \quad (k^2 = 3l^2), \tag{4.36}$$

we deduce from (4.1) that

$$K\delta\dot{A}_{111} = \epsilon\delta A_{111} - aB\delta A_{111} - PB^2\delta A_{111} - SA^2\delta A_{111}. \tag{4.37}$$

For large values of $a^2\epsilon^{-1}$ this equation reduces to

$$K\delta\dot{A}_{111} = -aB\delta A_{111}.$$

Thus the question as to the stability of the solution (4.35) is reduced to an investigation of the stability of the two-dimensional motion which was discussed above. We therefore conclude that, if for small values of ϵ a stationary solution of the form (4.35) exists (i.e. A and B are real), this solution is unstable for large values of $a^2\epsilon^{-1}$.

We have assumed above that $a^2\epsilon^{-1}$ is sufficiently large, without specifying any lower limit. The question arises whether it suffices to consider values of $a^2\epsilon^{-1}$ given by (3.7). No attempt has been made to investigate this matter. It should be remembered in this connexion that the boundary conditions applied here are somewhat artificial, and the findings above are therefore mainly of qualitative value.

5. On the stability of hexagons

Even if hexagons are stable for the special disturbances δA_{111} and δA_{021}, they may be unstable for other possible disturbances. Since we have assumed throughout the paper that $\Delta\beta$ is small (i.e. the amplitude of the convection is small), it is evident that it is only necessary to take into account disturbances satisfying (1.1).

The solution corresponding to hexagons may be written

$$A_{111}\cos kx\cos ly\sin\lambda z + A_{021}\cos 2ly\sin\lambda z, \tag{5.1}$$

where $k^2 = 3l^2$ and $A_{111} = 2A_{021}$. According to the above, a necessary condition in order that this solution be stable is that (1.21) be fulfilled. There are two classes of disturbances which we must take into account, viz. disturbances where $k^2 = 3l^2$ and disturbances where $k^2 \neq 3l^2$ (denoted by k' and l'). We will first investigate the former disturbances.

Let us consider the disturbance

$$(\delta A_{111} \cos kx \cos ly + \delta A_{021} \cos 2ly + \delta B_{111} \cos kx \sin ly + \delta B_{021} \sin 2ly) \sin \lambda z. \quad (5.2)$$

The equations for the amplitudes δA_{111} and δA_{021} are found by introducing

$$A_{111} + \delta A_{111}, \quad A_{021} + \delta A_{021} \quad (5.3)$$

into equations (1.8), (1.9) and applying the equilibrium equations. This gives

$$K\delta A_{111} = -2RA_{111}^2 \delta A_{111} - (a + 2PA_{021}) A_{111} \delta A_{021}, \quad (5.4)$$

$$K\delta A_{021} = -\tfrac{1}{2}(a + 2PA_{021}) A_{111} \delta A_{111} + (\tfrac{1}{4}aA_{111}^2/A_{021} - 2R_1 A_{021}^2) \delta A_{021}. \quad (5.5)$$

These are the equations that were discussed by Segel & Stuart and commented on in § 3. We assume in what follows that the motion is stable for these perturbations, i.e. that (2.1) is fulfilled. The equations governing δB_{111} and δB_{021}, are found by comparison with the derivation of the equations (1.8), (1.9) (see Palm 1960) to be

$$K\delta B_{111} = (\epsilon + aA_{021} - RA_{111}^2 - PA_{021}^2) \delta B_{111} - aA_{111} \delta B_{021}, \quad (5.6)$$

$$K\delta B_{021} = -\tfrac{1}{2}aA_{111} \delta B_{111} + (\epsilon - R_1 A_{021}^2 - \tfrac{1}{2}PA_{111}^2) \delta B_{021}, \quad (5.7)$$

where higher-order terms in δB_{111} and δB_{021} are neglected. Because A_{111} and A_{021} are equilibrium solutions of (1.8), (1.9), the equations reduce to

$$K\delta B_{111} = 2aA_{021} \delta B_{111} - \tfrac{1}{4}A_{111} \delta B_{021}, \quad (5.8)$$

$$K\delta B_{021} = -\tfrac{1}{2}aA_{111} \delta B_{111} + \tfrac{1}{4}aA_{111}^2/A_{021} \delta B_{021}. \quad (5.9)$$

We notice that the determinant of this system is zero. Hence, it may be concluded that if only linear terms are retained, the solution corresponding to hexagons is *neutral* for disturbances of the form (5.2).

It should here be emphasized that in many practical problems the occurrence of neutral (*linear*) disturbances involves practical instability, since a slight change in the coefficients may lead to instability. It does not seem reasonable that this is true in the actual case. A change in the coefficients in the equilibrium equations—due to a higher degree of approximation—leads to a corresponding change in the coefficients in the perturbation equations. This is, of course, due to the fact that the actual disturbance is very much of the same form as the equilibrium solution. To decide whether the motion is stable or not, we have therefore to take into account non-linear terms in δB_{111} and δB_{021}. It is readily shown that the next approximation is obtained by adding the terms

$$-R\delta B_{111}^3 - P\delta B_{021}^2 \delta B_{111} \quad (5.10)$$

on the right in (5.6), and the terms

$$-R_1 \delta B_{021}^3 - \tfrac{1}{2}P\delta B_{111}^2 \delta B_{021} \quad (5.11)$$

on the right in (5.7). Since all these terms are stabilizing, it seems reasonable to draw the conclusion that the motion is stable for the disturbance (5.2). This may be shown more rigorously by introducing

$$\delta u = \delta B_{111} + 2\delta B_{021}, \tag{5.12}$$

$$\delta v = \delta B_{111} - \delta B_{021}, \tag{5.13}$$

and applying the condition that $A_{111} = 2A_{021}$. The equations then take the form

$$K\delta\dot{u} = -\tfrac{1}{27}(R + 2P + 2R_1)\,\delta u^3 + \ldots, \tag{5.14}$$

$$K\delta\dot{v} = 3aA_{021}\,\delta v + \ldots, \tag{5.15}$$

where ... in (5.14) stands for terms of the form $\delta u^2\,\delta v$, $\delta u\,\delta v^2$, δv^3 and higher-order terms, and in (5.15) for higher-order terms. Since aA_{021} is negative, δv will decay and δu^3 becomes the important term in (5.14). The coefficient of this term is negative; hence the motion is stable. (This may also be shown by constructing a Liapunov function, see La Salla & Lefschetz 1961, pp. 49–51.) Correspondingly, it may be shown that the motion is stable for perturbations where $\cos kx$ is replaced by $\sin kx$.

To examine the stability of the second class of disturbances, it suffices to investigate a disturbance of the form

$$\delta A'_{111} \cos k'x \cos ly' \sin \lambda z. \tag{5.16}$$

It is found from (2.5) that $\delta A'_{111}$ is determined by

$$K\delta\dot{A}'_{111} = \epsilon\delta A'_{111} - SA^2_{111}\delta A'_{111} - TA^2_{021}\delta A'_{111}. \tag{5.17}$$

It should here be noted that (5.17) holds true even if other disturbances are present since no coupling terms exist between $\delta A'_{111}$ and possible other disturbances (we consider here only linear disturbances). Introducing for ϵ the value found from the equilibrium equations, we may write (5.17) as

$$K\delta\dot{A}'_{111} = [aA_{021} + (R-S)A^2_{111} + (P-T)A^2_{021}]\,\delta A'_{111}. \tag{5.18}$$

Since aA_{021} is supposed to be negative, this term is stabilizing. Because $A_{111} = 2A_{021}$, the two other terms on the right are also stabilizing if

$$4S + T - (4R + P) > 0. \tag{5.19}$$

If we write

$$k' = r\cos\phi, \tag{5.20}$$

$$l' = r\sin\phi, \tag{5.21}$$

then (5.19) is a function of ϕ. A numerical examination of the terms reveals that (5.19) is satisfied for all values of ϕ. Actually, with a very high degree of accuracy, the left-hand side in (5.19) is independent of ϕ and equal to R_1. We therefore arrive at the conclusion that the solution corresponding to hexagons is stable for this class of disturbances when (1.21) is fulfilled. It should perhaps be pointed out that the calculation reveals that the important stabilizing agency is the (vertical) temperature convection.

6. Conclusions

It seems clear, from previous papers (Palm 1960; Segel & Stuart 1962) and the present one, that the effect of the variation of kinematic viscosity with temperature is most important as regards the final form of the cellular motion in thermal convection.

The present paper is restricted to considering thermal convection where (i) the overcritical temperature gradient $\Delta\beta$ is small enough for it to be sufficient to take into account only third-order terms, and (ii) the relative variation of the kinematic viscosity in the actual temperature region is small, i.e. γ/ν_0 is small compared to unity. Applying these assumptions, we believe we have proved that, when the condition found by Segel & Stuart (2.1) is satisfied, the solution corresponding to a hexagonal pattern is stable for all small disturbances which can possibly occur. To prove this, non-linear disturbance theory had to be taken into account.

The other problem, of whether this solution is the *only* stable one when $a^2\epsilon^{-1}$ is sufficiently large, seems to be far more complicated. We have been able to show that the solution corresponding to hexagons is the only stable one when the motion consists of the three wave-components (1.3), (1.6) and (2.4). It should be emphasized that this result may easily be extended to comprehend n wave-components in addition to A_{111} and A_{021}, provided the n wave-components do not interact in such a way that they give rise to terms proportional to a. This last restriction means that the group of waves considered defines only *one* system of hexagons. However, for groups of wave-components defining more than one system of hexagons, rotated in relation to each other, the problem is considerably more involved. In order to investigate this case, it would be reasonable first to study the stability of a group of wave-components which define *two* systems of hexagons, for example the following group:

$$A\cos(kx+ly)\sin\lambda z, \tag{6.1}$$

$$B\cos(kx-ly)\sin\lambda z, \tag{6.2}$$

$$C\cos 2ly\sin\lambda z; \tag{6.3}$$

$$A'\cos(k'x+l'y)\sin\lambda z, \tag{6.4}$$

$$B'\cos(k''x+l''y)\sin\lambda z, \tag{6.5}$$

$$C'\cos[(k'-k'')x+(l'-l'')y]\sin\lambda z; \tag{6.6}$$

where

$$k' = r\cos\phi_0,$$

$$l' = r\sin\phi_0,$$

$$k'' = r\cos(\phi_0-\tfrac{1}{3}\pi),$$

$$l'' = r\sin(\phi_0-\tfrac{1}{3}\pi),$$

with ϕ_0 denoting an arbitrary angle. The first three wave-components (6.1), (6.2), (6.3), obviously lead to the system of hexagons discussed above, whereas the last three wave-components define a system of hexagons rotated in relation to the first one. These six wave-components must therefore contain at least two stable equilibrium solutions. Which of these will be realized depends on the initial conditions. If in this case no other equilibrium solutions are found to be stable, it seems very reasonable to draw the conclusion that hexagons are the *only* stable mode when all possible wave-components are taken into account. No serious attempt has, however, been made to solve this problem.

A hexagonal pattern is observed in experiments, a fact which must be interpreted to mean that the hexagonal solution is under certain conditions the only stable one. It seems very reasonable to suppose that in the case of realistic boundary conditions, $a^2\epsilon^{-1}$ must be larger than a certain value in order that this solution may be observed. Writing $\Delta\nu$ instead of γ, we should therefore expect that

$$\left(\frac{\Delta\nu}{\nu_0}\right)^2 \Big/ \left(\frac{\Delta\beta}{\beta_0}\right) \tag{6.7}$$

must be larger than a certain number, f say, in order that hexagons be observed. It seems reasonable to suppose that f is a function of the Prandtl number,

$$P = \nu/\kappa, \tag{6.8}$$

only, which is consistent with (3.7). The condition for observing hexagons should then be

$$\left(\frac{\Delta\nu}{\nu_0}\right)^2 \Big/ \left(\frac{\Delta\beta}{\beta_0}\right) > f(P), \tag{6.9}$$

where f depends only on the boundary conditions. To the author's knowledge, no experimental results concerning this problem have yet been published.

REFERENCES

CHANDRASEKHAR, C. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

GANTMACHER, F. R. 1959 *Applications for the Theory of Matrices*, English edn, trans. by J. L. Brenner, D. W. Bushaw & S. Evanusa. New York: Interscience.

LA SALLA, J. & LEFSCHETZ, S. 1961 *Stability by Liapunov's Direct Method*. New York: Academic Press.

PALM, E. 1960 *J. Fluid Mech.* **8**, 183.

SEGEL, L. A. & STUART, J. T. 1962 *J. Fluid Mech.* **13**, 289.